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Digital Logic Design

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Boolean Algebra: Axioms

Formal version	English version
1. B contains at least two elements, 0 and 1, such that $0 \neq 1$	Math formality…
2. *Closure* $a,b \in B$, (i) $a + b \in B$ (ii) $a \bullet b \in B$	Result of AND, OR stays in set you start with
3. *Commutative Laws*: $a,b \in B$, (i) $a + b = b + a$ (ii) $a \bullet b = b \bullet a$	For primitive AND, OR of 2 inputs, order doesn't matter
4. *Identities*: $0, 1 \in B$ (i) $a + 0 = a$ (ii) $a \bullet 1 = a$	There are identity elements for AND, OR, that give you back what you started with
5. Associativity (i) $a + (b + c) = (a + b) + c$ (ii) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$	For the same operation, parenthesis order does not matter
5. *Distributive Laws*: (i) $a + (b \bullet c) = (a + b) \bullet (a + c)$ (ii) $a \bullet (b + c) = a \bullet b + a \bullet c$	• distributes over +, just like algebra …but + distributes over •, also (!!)
6. *Complement*: (i) $a + \bar{a} = 1$ (ii) $a \bullet \bar{a} = 0$	There is a complement element; AND/ORing with it gives the identity elm.

Boolean Algebra: Useful Laws

Dual

Operations with 0 and 1:

1. $X + 0 = X$ — 1D. $X \bullet 1 = X$
2. $X + 1 = 1$ — 2D. $X \bullet 0 = 0$

AND, OR with identities gives you back the original variable or the identity

Idempotent Law:

3. $X + X = X$ — 3D. $X \bullet X = X$

AND, OR with self = self

Involution Law:

4. $\overline{(\overline{X})} = X$

double complement = no complement

Laws of Complementarity:

5. $X + \overline{X} = 1$ — 5D. $X \bullet \overline{X} = 0$

AND, OR with complement gives you an identity

Commutative Law:

6. $X + Y = Y + X$ — 6D. $X \bullet Y = Y \bullet X$

Just an axiom…

Useful Laws

- Useful for simplifying expressions
- Actually worth remembering
  - They show up a lot in real designs…

1. $X \bullet Y + X \bullet \overline{Y} = X$
2. $X + X \bullet Y = X$
3. $(X + \overline{Y}) \bullet Y = X \bullet Y$

1D. $(X + Y) \bullet (X + \overline{Y}) = X$

2D. $X \bullet (X + Y) = X$

3D. $(X \bullet \overline{Y}) + Y = X + Y$

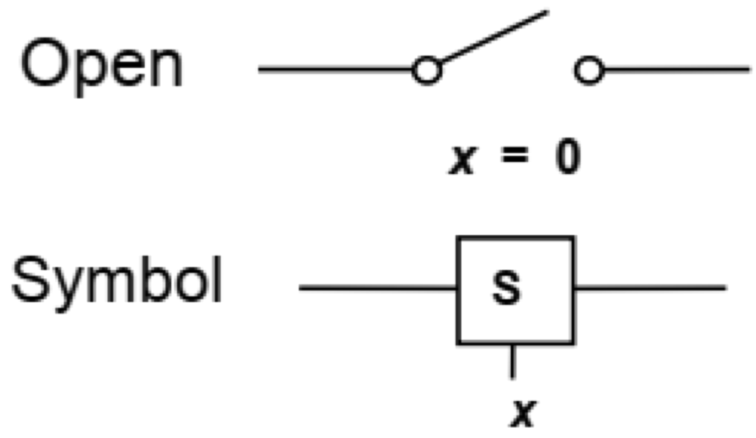
Outline

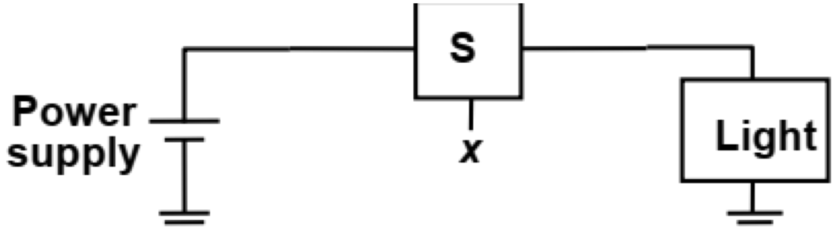
- Using Boolean Equations to Represent a Logic Circuit
- Switching Function

Boolean Equations for Representing a Logic Circuit

Switches

- A switch has two states
  - Closed/ On
  - Open/ OFF





Switching Algebra

- Switching algebra
  - Boolean algebra with the set of elements $K = \{0, 1\}$
- Switching Function
  - If there are *n* variables, we can define 2^{2^n} switching functions.
- Sixteen functions of two variables

AB	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}
00	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
01	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
10	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
11	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

Representing Switching Functions

- Table
  - **Truth table** is a **unique signature** of a **Boolean function**
  - It is an expensive representation
- Switching expressions
  - $f_0(A,B) = 0$
  - $f_6(A,B) = AB' + A'B$
  - $f_{11}(A,B) = AB + A'B + A'B' = A' + B$, ...

AB	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}
00	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
01	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
10	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
11	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

Value of Switching Functions

- Plugging in the values of all variables:
  - Value of f_6 when A = 1 and B = 0 is: $1.0' + 1'.0 = 1 + 0 = 1$.

AB	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}
00	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
01	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
10	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
11	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

Some Definition: 1

- **Variable**
  - Operand
  - $\boldsymbol{A}$, **B**
- **Complement**
  - A variable with a bar over it
  - $\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}}$
- **Literal**
  - A variable, complemented or uncomplemented
  - $\boldsymbol{A}$, $\overline{\boldsymbol{A}}$, **B** , $\overline{\boldsymbol{B}}$
- **Implicant**
  - $(\boldsymbol{A} \cdot \boldsymbol{B} \cdot \overline{\boldsymbol{C}})$, $(\overline{\boldsymbol{A}} \cdot \boldsymbol{C})$, $(\boldsymbol{B} \cdot \overline{\boldsymbol{C}})$

Some Definition: 1

- **Product term**
  - A literal or literals ANDed together
  - $(A \cdot B \cdot \overline{C})$, $(\overline{A} \cdot C)$, $(B \cdot \overline{C})$
- **Sum term**
  - A literal or literals ORed together.
  - $(A + B + \overline{C})$, $(\overline{A} + C)$, $(B + \overline{C})$
- **Minterm**
  - A product that includes all the variables
  - $(A \cdot B \cdot \overline{C})$, $(\overline{A} \cdot \overline{B} \cdot C)$, $(\overline{A} \cdot B \cdot \overline{C})$
- **Maxterm**
  - A sum that includes all the variables
  - $(A + B + \overline{C})$, $(\overline{A} + \overline{B} + C)$, $(\overline{A} + B + \overline{C})$

Minterms

- A product that includes all the variables
  - $(\boldsymbol{A \cdot B \cdot \overline{C}})$, $(\boldsymbol{\overline{A} \cdot \overline{B} \cdot C})$, $(\boldsymbol{\overline{A} \cdot B \cdot \overline{C}})$
  - Each row in a truth table has a minterm
  - A minterm is a product (AND) of literals
  - Each minterm is TRUE for that row (and only that row)

- Minterms of three variables:

Minterm	Minterm Code	Minterm Number
$A'B'C'$	000	m_0
$A'B'C$	001	m_1
$A'BC'$	010	m_2
$A'BC$	011	m_3
$AB'C'$	100	m_4
$AB'C$	101	m_5
ABC'	110	m_6
ABC	111	m_7

Algebraic Forms Using Products

- **SOP**
  - Sum of Products
  - ORing product terms
  - f(A, B, C) = ABC + A'C + B'C
  - f(A, B, C) = ABC + A'CB + A'CB' + B'C

- **Canonical SOP**
  - Represented as a sum of **minterms** only.
  - Example: $f_1(A,B,C) = A'BC' + ABC' + A'BC + ABC$

SOP

- **All Boolean equations can be written in SOP form**
  - A.k.a., disjunctive normal form or minterm expansion
  - Find all the input combinations (minterms) for which the output of the function is TRUE.

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

$$F = \underbrace{\bar{A}BC}_{011} + \underbrace{A\bar{B}\bar{C}}_{100} + \underbrace{A\bar{B}C}_{101} + \underbrace{AB\bar{C}}_{110} + \underbrace{ABC}_{111}$$

Why Does SOP Work?

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

This input

$$\overset{0\,1\,1}{\bar{A}BC} \quad \overset{1\,0\,0}{A\bar{B}\bar{C}} \quad \overset{1\,0\,1}{A\bar{B}C} \quad \overset{1\,1\,0}{AB\bar{C}} \quad \overset{1\,1\,1}{ABC}$$

$$F = \bar{A}BC + A\bar{B}\bar{C} + A\bar{B}C + AB\bar{C} + ABC$$

Activates this term

- **Only** the **shaded product term**
  - $A\bar{B}C = 1 \cdot \bar{0} \cdot 1$ will be 1
- **No** other product terms will “turn on” — they will all be 0
- So if inputs A B C correspond to a product term in expression,
  - We get 0 + 0 + … + 1 + … + 0 + 0 = 1 for output
- If inputs A B C do not correspond to any product term in expression
  - We get 0 + 0 + … + 0 = 0 for output

Representing Canonical SOP

- Compact form
  - $f_1(A,B,C) = A'BC' + A'BC + ABC' + ABC$
  - $f_1(A,B,C) = m_2 + m_3 + m_6 + m_7$
- A further simplified form
  - $f_1(A,B,C) = \sum m\ (2,3,6,7)$ (minterm list form)

Minterm	Minterm Code	Minterm Number
$A'B'C'$	000	m_0
$A'B'C$	001	m_1
$A'BC'$	010	m_2
$A'BC$	011	m_3
$AB'C'$	100	m_4
$AB'C$	101	m_5
ABC'	110	m_6
ABC	111	m_7

Notation for SOP

- Standard "shorthand" notation
  - If we agree on the order of the variables in the rows of truth table…
  - Then, we can enumerate each row with the decimal number that corresponds to the binary number created by the input pattern

A	B	C	F	
0	0	0	0	
0	0	1	0	
0	1	0	0	
0	1	1	1	
1	0	0	1	100 = decimal 4 so this is minterm #4, or m4
1	0	1	1	
1	1	0	1	
1	1	1	1	111 = decimal 7 so this is minterm #7, or m7

f =

We can write this as a sum of products

Or, we can use a summation notation

Important Rules in Compact Forms!

- Order of variables in the functional notation is important
  - m_1 = A'BC'
  - m_4 = BA'C'

- $\sum_{i=0}^{2^n-1} m_i = 1$
  - $AB + (AB)' = 1$
  - $AB + A' + B' = 1$
  - $AB + A'B' \neq 1$.

Minterm	Minterm Code	Minterm Number
$A'B'C'$	000	m_0
$A'B'C$	001	m_1
$A'BC'$	010	m_2
$A'BC$	011	m_3
$AB'C'$	100	m_4
$AB'C$	101	m_5
ABC'	110	m_6
ABC	111	m_7

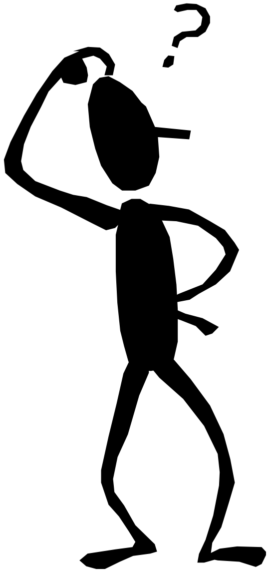
Truth Table & Minterms

- Deriving truth table of $f_1(A,B,C)$ from minterm list:
  - $f_1(A,B,C) = A'BC' + A'BC + ABC' + ABC$
  - $f_1(A,B,C) = m_2 + m_3 + m_6 + m_7$

Row No. (i)	**Inputs** ABC	**Outputs** $f_1(A,B,C)=\Sigma m(2,3,6,7)$	**Complement** $f_1'(A,B,C)=\Sigma m(0,1,4,5)$
0	000	0	1 $\leftarrow$ m_0
1	001	0	1 $\leftarrow$ m_1
2	010	1 $\leftarrow$ m_2	0
3	011	1 $\leftarrow$ m_3	0
4	100	0	1 $\leftarrow$ m_4
5	101	0	1 $\leftarrow$ m_5
6	110	1 $\leftarrow$ m_6	0
7	111	1 $\leftarrow$ m_7	0

SOP: Example

- Suppose $f(A,B,Q,Z) = A'B'Q'Z' + A'B'Q'Z + A'BQZ' + A'BQZ$
- **Express $f(A,B,Q,Z)$ and $f\,'(A,B,Q,Z)$ in minterm list form.**



SOP: Example (cont'd)

$f(A,B,Q,Z) = A'B'Q'Z' + A'B'Q'Z + A'BQZ' + A'BQZ$

$= m_0 + m_1 + m_6 + m_7$

$= \Sigma\, m(0, 1, 6, 7)$

$f\,'(A,B,Q,Z) = m_2 + m_3 + m_4 + m_5 + m_8 + m_9 + m_{10} + m_{11} + m_{12}$

$+ m_{13} + m_{14} + m_{15}$

$= \Sigma\, m(2, 3, 4, 5, 8, 9, 10, 11, 12, 13, 14, 15)$

Canonical SOP Forms

A	B	C	minterms
0	0	0	$\bar{A}\bar{B}\bar{C}$ = m0
0	0	1	$\bar{A}\bar{B}C$ = m1
0	1	0	$\bar{A}B\bar{C}$ = m2
0	1	1	$\bar{A}BC$ = m3
1	0	0	$A\bar{B}\bar{C}$ = m4
1	0	1	$A\bar{B}C$ = m5
1	1	0	$AB\bar{C}$ = m6
1	1	1	ABC = m7

Shorthand Notation for Minterms of 3 Variables

F in canonical form:

$$F(A,B,C) = \sum m(3,4,5,6,7)$$
$$= m3 + m4 + m5 + m6 + m7$$

$F =$

canonical form ≠ minimal form

F

Maxterm

- A sum that includes all the variables
  - $(\boldsymbol{A} + \boldsymbol{B} + \overline{\boldsymbol{C}})$, $(\overline{\boldsymbol{A}} + \overline{\boldsymbol{B}} + \boldsymbol{C})$, $(\overline{\boldsymbol{A}} + \boldsymbol{B} + \overline{\boldsymbol{C}})$

- Maxterms of three variables:

Maxterm	Maxterm Code	Maxterm Number
$A+B+C$	000	M_0
$A+B+C'$	001	M_1
$A+B'+C$	010	M_2
$A+B'+C'$	011	M_3
$A'+B+C$	100	M_4
$A'+B+C'$	101	M_5
$A'+B'+C$	110	M_6
$A'+B'+C'$	111	M_7

Algebraic Forms Using Sums!

- **POS**
  - Product of Sums
  - ANDing sum terms
  - f (A, B, C) = (A' + B' + C')(A + C')(B + C')

- **Canonical POS**
  - Represented as a product of **maxterms** only.
  - Example: $f_2(A,B,C) = (A+B+C)(A+B+C')(A'+B+C)(A'+B+C')$

POS

We can have another from of representation

DeMorgan of SOP of $\overline{F}$

$$F = (A + B + C)(A + B + \overline{C})(A + \overline{B} + C)$$

products · sums

Each sum term represents one of the "zeros" of the function

This input

$$\begin{matrix} 0 \ 0 \ 0 & 0 \ 0 \ 1 & 0 \ 1 \ 0 \\ F = (A + B + C) & (A + B + \overline{C}) & (A + \overline{B} + C) \end{matrix}$$

Activates this term

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

For the given input, only the shaded sum term will equal 0

$$A + \overline{B} + C = 0 + \overline{1} + 0$$

Anything ANDed with 0 is 0; Output F will be 0

Representing Canonical POS

- $f_2(A,B,C) = (A+B+C)(A+B+C')(A'+B+C)(A'+B+C')$
- $f_2(A,B,C) = M_0M_1M_4M_5$

  $= \Pi M(0,1,4,5)$ (maxterm list form)
- Truth table for $f_2(A,B,C)$:

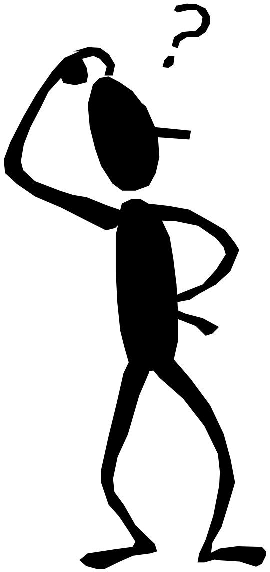
Rwo No. (i)	**Inputs** ABC	M_0 $A+B+C$	M_1 $A+B+C'$	M_4 $A'+B+C$	M_5 $A'+B+C'$	**Outputs** $f_2(A,B,C)$
0	000	0	1	1	1	0
1	001	1	0	1	1	0
2	010	1	1	1	1	1
3	011	1	1	1	1	1
4	100	1	1	0	1	0
5	101	1	1	1	0	0
6	110	1	1	1	1	1
7	111	1	1	1	1	1

Sample 1

- Present using minterms and maxterms

$f_1(A,B,C) = A'BC' + A'BC + ABC' + ABC$

$f_2(A,B,C) = (A+B+C)(A+B+C')(A'+B+C)(A'+B+C')$



Sample 1

- $f_1(A,B,C) = A'BC' + A'BC + ABC' + ABC = m_2 + m_3 + m_6 + m_7$
- $f_2(A,B,C) = (A+B+C)(A+B+C')(A'+B+C)(A'+B+C') = M_0M_1M_4M_5$
- Truth tables of $f_1(A,B,C)$ and $f_2(A,B,C)$ are **identical**.
- $f_1(A,B,C) = \Sigma\, m\,(2,3,6,7)$

  $= \Pi M(0,1,4,5)$

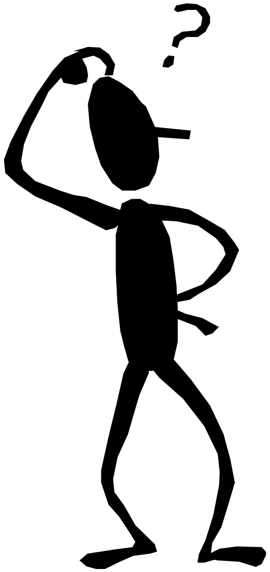
  $= f_2(A,B,C)$

A	B	C	f_1	f_2
0	0	0	0	0
0	0	1	0	0
0	1	0	1	1
0	1	1	1	1
1	0	0	0	0
1	0	1	0	0
1	1	0	1	1
1	1	1	1	1

Sample 2

- Construct the truth table and express in both maxterm and minterm form.

$$f(A,B,C) = (A+B+C')(A+B'+C')(A'+B+C')(A'+B'+C')$$



Sample 2

- $f(A,B,C) = (A+B+C')(A+B'+C')(A'+B+C')(A'+B'+C')$
  - $f(A,B,C) = M_1M_3M_5M_7 = \Pi M(1,3,5,7) = \Sigma\, m\,(0,2,4,6)$

Row No. (i)	**Inputs** ABC	**Outputs** $f(A,B,C) = \Pi\, M(1,3,5,7) = \Sigma m(0,2,4,6)$
0	000	1 m_0
1	001	0 $\leftarrow$ M_1
2	010	1 m_2
3	011	0 $\leftarrow$ M_3
4	100	1 m_4
5	101	0 $\leftarrow$ M_5
6	110	1 m_6
7	111	0 $\leftarrow$ M_7

Minterms VS. Maxterms

- Relationship between minterm m_i and maxterm M_i

- For $f(A,B,C)$, consider $m_1 = A'B'C$
  - $(m_1)' = (A'B'C)' = A + B + C' = M_1$

- In general, $(m_i)' = M_i$

  $(Mi)' = ((m_i)')' = m_i$

Minterms VS. Maxterms (cont'd)

- Consider the below function and its complement
- f(A,B,C) = (A+B+C')(A+B'+C')(A'+B+C')(A'+B'+C')
  - f(A,B,C) = ΠM(1,3,5,7)
  - f '(A,B,C) = ΠM(0,2,4,6)

Row No. (i)	**Inputs** ABC	**Outputs** $f(A,B,C)$	**Outputs** $f'(A,B,C) = \Pi M(0,2,4,6)$
0	000	1	0 ← M_0
1	001	0	1
2	010	1	0 ← M_2
3	011	0	1
4	100	1	0 ← M_4
5	101	0	1
6	110	1	0 ← M_6
7	111	0	1

Sample 3

- $f(A,B,C) = (A+B+C')(A+B'+C')(A'+B+C')(A'+B'+C')$
  - $f(A,B,C) = \Pi M(1,3,5,7)$
  - $f'(A,B,C) = \Pi M(0,2,4,6)$

- $f(A,B,C) \cdot f'(A,B,C) = ?$

Sample 3

- f(A,B,C) = (A+B+C')(A+B'+C')(A'+B+C')(A'+B'+C')
  - f(A,B,C) = ΠM(1,3,5,7)
  - f '(A,B,C) = ΠM(0,2,4,6)

- $f(A,B,C) \cdot f'(A,B,C) = 0$
  - $(M_0 M_2 M_4 M_6)(M_1 M_3 M_5 M_7) = 0$

- In general
  - $f(A,B,C) = \Sigma\, m\,(0,2,4,6) = \Pi M(1,3,5,7)$
  - $f'(A,B,C) = \Sigma\, m\,(1,3,5,7) = \Pi M(0,2,4,6)$

$$\prod_{i=0}^{2^n-1} M_i = 0$$

How to Present in Canonical SOP, POS?

- **Derive canonical POS or SOP using switching algebra.**

- Use Theorem 6 to add missing literals.
  - $AB + AB' = A$
  - $(A+B)(A+B') = A$

- Example: $f(A,B,C) = AB + AC' + A'C$ to canonical SOP form.
  - $AB = ABC' + ABC = m_6 + m_7$
  - $AC' = AB'C' + ABC' = m_4 + m_6$
  - $A'C = A'B'C + A'BC = m_1 + m_3$
  - $f(A,B,C) = (m_6 + m_7) + (m_4 + m_6) + (m_1 + m_3) = \Sigma m(1, 3, 4, 6, 7)$

- Example: $f(A,B,C) = A(A + C')$ to canonical POS form.
  - $A = (A+B')(A+B) = (A+B'+C')(A+B'+C)(A+B+C')(A+B+C)$
    $= M_3M_2M_1M_0$
  - $(A+C')= (A+B'+C')(A+B+C') = M_3M_1$
  - $f(A,B,C) = (M_3M_2M_1M_0)(M_3M_1) = \Pi M(0, 1, 2, 3)$

How to Present in Canonical SOP, POS? (cont'd)

- **Derive canonical POS or SOP using switching algebra.**

- Theorem 10: Shannon's expansion theorem
  - $f(x_1, x_2, \ldots, x_n) = x_1\, f(1, x_2, \ldots, x_n) + (x_1)'\, f(0, x_2, \ldots, x_n)$
  - $f(x_1, x_2, \ldots, x_n) = [x_1 + f(0, x_2, \ldots, x_n)]\, [(x_1)' + f(1, x_2, \ldots, x_n)]$

- Example: $f(A,B,C) = AB + AC' + A'C$
  - $f(A,B,C) = AB + AC' + A'C = A\, f(1,B,C) + A'\, f(0,B,C)$
    $= A(1\times B + 1\times C' + 1'\times C) + A'(0\times B + 0\times C' + 0'\times C) = A(B + C') + A'C$
  - $f(A,B,C) = A(B + C') + A'C = B[A(1+C') + A'C] + B'[A(0 + C') + A'C]$
    $= B[A + A'C] + B'[AC' + A'C] = AB + A'BC + AB'C' + A'B'C$
  - $f(A,B,C) = AB + A'BC + AB'C' + A'B'C$
    $= C[AB + A'B\times 1 + AB'\times 1' + A'B'\times 1] + C'[AB + A'B\times 0 + AB'\times 0' + A'B'\times 0]$
    $= ABC + A'BC + A'B'C + ABC' + AB'C'$

Incompletely Specified Functions

- A switching function may be incompletely specified.
- Some minterms are omitted, which are called don't-care minterms.
- Don't cares arise in two ways:
  - Certain input combinations never occur.
  - Output is required to be 1 or 0 only for certain combinations.
  - Don't care minterms: d_i
  - Don't care maxterms: D_i
- Example: f(A,B,C) has minterms m_0, m_3, and m_7 and don't-cares d_4 and d_5.
  - Minterm list is: $f(A,B,C) = \Sigma m(0,3,7) + d(4,5)$
  - Maxterm list is: $f(A,B,C) = \Pi M(1,2,6) \cdot D(4,5)$
  - $f'(A,B,C) = \Sigma m(1,2,6) + d(4,5) = \Pi M(0,3,7) \cdot D(4,5)$
  - $f(A,B,C) = A'B'C' + A'BC + ABC + d(AB'C' + AB'C)$
  - $= B'C' + BC$ (use d_4 and omit d_5)

Thank You