

Iran University of Science & Technology

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Digital Logic Design

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Signed Numbers Representation





- Let N = $(a_{n-1} \dots a_0)_2$
 - If $N \ge 0$, it is represented by $(0a_{n-1} \dots a_0)_2$
 - If N < 0, it is represented by $[0a_{n-1} \dots a_0]_2$
 - $[N]_2 = 2^n (N)_2$
- Diminished radix complement [N]_{r-1}
 - $[N]_{r-1} = r^n (N)_r 1$
- One's complement (r = 2):
 - $[N]_{2-1} = 2^n (N)_2 1$



Overflow Condition

- Presenting numbers using two's complement number system:
 - Addition: Add two numbers.
 - Subtraction: Add two's complement of the subtrahend to the minuend.
 - Carry bit is discarded
 - **Overflow** is detected as the Table.

Case	Carry	Sign Bit	Condition	Overflow ?
B + C	0	0	$\mathbf{B} + \mathbf{C} \le 2^{n-1} - 1$	No
	0	1	$B + C > 2^{n-1} - 1$	Yes
B - C	1	0	$B \leq C$	No
	0	1	B > C	No
-B - C	1	1	$-(\mathbf{B} + \mathbf{C}) \ge -2^{n-1}$	No
	1	0	$-(B+C) < -2^{n-1}$	Yes



Outline

• Digital codes



Computer Codes



Computer Codes

• Code

- Systematic use of a given set of symbols
- => for representing information

Numeric Codes

- To represent numbers
 - Fixed-point numbers
 - Floating-point number



Color	3-bit code
Red	000
Orange	001
Yellow	010
Green	011
Blue	100
Indigo	101
Violet	110



Computer Codes (cont'd)

• Fixed-point Numbers

- Used for signed integers or integer fractions
- Sign magnitude, two's complement, or one's complement systems are used.
- Integer: (Sign bit) + (Magnitude) + (Implied radix point)
- Fraction: (Sign bit) + (Implied radix point) + (Magnitude)





Computer Codes (cont'd)

- Excess or Biased Representation
 - Excess-K representation of a code C:
 - Add K to each code word C.
 - Used for exponents of floating-point numbers
- Example
 - Excess-8 representation of 4-bit two's complement code

TABLE 1.8 EXCESS-8 CODE

Decimal	Two's Complement	Excess-8
+7	0111	1111
+6	0110	1110
+5	0101	1101
+4	0100	1100
+3	0011	1011
+2	0010	1010
+1	0001	1001
0	0000	1000
-1	1111	0111
-2	1110	0110
-3	1101	0101
-4	1100	0100
-5	1011	0011
-6	1010	0010
-7	1001	0001
-8	1000	0000



Characters and Other Codes

- To represent information as strings of alpha-numeric characters
 - Binary coded decimal (BCD)
 - ASCII
 - Gray code
 - Error detection codes
 - Error correction codes
 - Hamming code



Binary Coded Decimal (BCD)

- Used to represent the decimal digits 0 9
- 4 bits are used.
- Each bit position has a weight associated with it
 - Weighted code
 - 8, 4, 2, and 1 from MSB to LSB (called 8-4-2-1 code)

•BCD Codes:

- 0:0000
- 1:0001
- 2:0010
- 3:0011
- 4:0100
- 5:0101
- 6:0110
- 7:0111
- 8: 1000
- 9: 1001



BCD in Applications!

- Used to encode numbers for output to numerical displays
- Used in processors that perform decimal arithmetic
- Example
 - $(9750)_{10} = (1001, 0111, 0101, 0000)_{BCD}$
 - $(9750)_{10} = (10010111010000)_{BCD}$



BCD: Sample

- $(9)_{10} = (?)_{BCD}$
- $(29)_{10} = ()_{BCD}$
- $(129)_{10} = ()_{BCD}$
- $(1029)_{10} = ()_{BCD}$





BCD: Sample (cont'd)

- $(9)_{10} = (1001)_{BCD}$
- $(29)_{10} = (0010\ 1001)_{BCD} = (00101001)_{BCD}$
- $(129)_{10} = (0001\ 0010\ 1001)_{BCD} = (000100101001)_{BCD}$
- $(1029)_{10} = (0001\ 0000\ 0010\ 1001)_{BCD} = (000100000101001)_{BCD}$



Gray Code

• Gray code:

- A cyclic code with the property
- Cyclic code: a circular shifting of a code word produces another code word
- Property: two consecutive code words differ in only 1 bit
 - Distance between the two code words is 1





Gray Code

• Bit i in gray code

- Compare bit i and i+1 in binary code
- Similar ==> Gray(i) = 0
- Otherwise ==> Gray(i) =1

Digit	Binary	Gray Code	Digit	Binary	Gray Code
0	0000	0000	0	1000	1100
1	0001	0001	1	1001	1101
2	0010	0011	2	1010	1111
3	0011	0010	3	1011	1110
4	0100	0110	4	1100	1010
5	0101	0111	5	1101	1011
6	0110	0101	6	1110	1001
7	0111	0100	7	1111	1000



Gray Code Conversion







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ASCII

- American Standard Code for Information Interchange
 - Most widely used character code
 - 7-bit ASCII code
 - Eighth bit is often used for error detection (parity bit)

					C ₆ C ₅	~ 4			
		0	1	2	3	4	5	6	7
	0	NUL	DLE	SP	0	0	Ρ	ì	р
	1	SOH	DC1 XON	!	1	А	Q	a	q
	2	STX	DC2	П	2	В	R	b	r
	з	ETX	DC3 XOFF	#	3	С	S	С	S
	4	EOT	DC4	\$	4	D	Т	d	t
	5	ENQ	NAK	%	5	E	U	е	u
C1 C0	6	ACK	SYN	&	6	F	V	f	\vee
-1-0	7	BEL	ETB	I	7	G	W	g	W
	8	BS	CAN	(8	Н	Х	h	x
	9	ΗT	EM)	9		Y	i	У
	Α	LF	SUB	*	:	J	Ζ	j	Z
	В	۲	ESC	+	;	К	[×	{
	С	FF	FS IS4	,	<	L	\		
	D	CR	GS IS3	-	=	М]	m	}
	Ε	S0	RS IS2		>	Ν	~	n	2
	F	SI	US IS1	/	?	0		0	DEL

 C_3C_2



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ASCII (cont'd)

- Control Character
 - First 32 characters of ASCII table are used for control
 - \circ 00 1F
 - One control character appears at end of ASCII table

 C_3C_2

• DEL

		C ₆ C ₅ C ₄							
		0	1	2	3	4	5	6	7
d	0	NUL	DLE	SP	0	0	Ρ	ì	р
	1	SOH	DC1 XON	!	1	А	Q	а	q
	2	STX	DC2	П	2	В	R	b	r
	3	ETX	DC3 X0FF	#	З	С	S	С	S
	4	EOT	DC4	\$	4	D	Т	d	t
	5	ENQ	NAK	%	5	Е	U	е	u
C1 C0	6	ACK	SYN	&	6	F	V	f	\vee
-1-0	7	BEL	ETB	1	7	G	W	g	W
	8	BS	CAN	(8	Н	Х	h	х
	9	HT	EM)	9		Υ	i	У
	Α	LF	SUB	*	:	J	Ζ	j	Z
	В	VΤ	ESC	+	;	К	[k	{
	С	FF	FS IS4	,	<	L	\		
	D	CR	GS IS3	-	=	М]	m	}
	Ε	S0	RS IS2		>	Ν	^	n	~
	F	SI	US IS1	/	?	0	_	0	DEL



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ASCII (cont'd)

• Example

 ASCII code representation of the word *Digital*

Digital					
Character	Binary Code				
D					
i					
g					
i					
t					
а					
I					

•					0.3	-		_	
		0	1	2	3	4	5	6	7
ord	0	NUL	DLE	SP	0	0	Ρ	,	a
	1	SOH	DC1 XON	!	1	А	Q	а	q
	2	STX	DC2	П	2	В	R	b	r
	3	ETX	DC3 XOFF	#	З	С	S	С	s
	4	EOT	DC4	\$	4	D	Т	d	t
	5	ENQ	NAK	%	5	Ш	С	е	u
C2C2C4C2	6	ACK	SYN	8	6	F	$^{\vee}$	f	\vee
-3-2-1-0	7	BEL	ETB	I	7	G	W	g	W
	8	BS	CAN	(8	Τ	Х	h	х
	9	ΗT	EM)	9	-	Y	i	У
	Α	Ľ	SUB	*		J	Ζ	j	Z
	В	VΤ	ESC	+	;	К	[k	{
	С	FF	FS IS4	,	<	L	\		
	D	CR	GS IS3	-	=	М]	m	}
	Ε	S0	RS IS2		>	Ν	^	n	~
	F	SI	US IS1	/	?	0	_	0	DEL

 $C_6C_5C_4$



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ASCII (cont'd)

• Example

 ASCII code representation of the word *Digital*

Digital					
Character	Binary Code				
D	100 0100				
i	110 1001				
g	110 0111				
i	110 1001				
t	111 0100				
а	110 0001				
I	110 1100				

					-6-5	-4			
		0	1	2	3	4	5	6	7
ord	0	NUL	DLE	SP	0	0	Ρ	ì	р
	1	SOH	DC1 XON		1	А	Q	a	q
	2	STX	DC2	н	2	В	R	b	r
	3	ETX	DC3 X0FF	#	3	С	S	С	S
	4	EOT	DC4	\$	4	D	Т	d	t
	5	ENQ	NAK	%	5	Е	U	е	u
$C_2C_2C_4C_6$	6	ACK	SYN	&	6	F	V	f	\vee
-3-2-1-0	7	BEL	ЕТВ	I	7	G	W	g	W
	8	BS	CAN	(8	Н	Х	h	x
	9	HT	EM)	9		Υ	i	У
	Α	LF	SUB	*	:	J	Ζ	j	z
	В	VΤ	ESC	+	;	К	[k	{
	С	FF	FS IS4		<	L	\		
	D	CR	GS IS3	-	=	М]	m	}
	Ε	S0	RS IS2		>	Ν	^	n	~
	F	SI	US IS1	/	?	0	_	0	DEL

CcCrCa



Character Codes

- Standard ASCII
 - 7-bit character codes (0 127)
- Extended ASCII
 - 8-bit character codes (0 255)
- Unicode
 - 16-bit character codes (0 65,535)
- Unicode standard represents a universal character set
 - Defines codes for characters used in all major languages
 - Used in Windows-XP: each character is encoded as 16 bits
- UTF-8: variable-length encoding used in HTML
 - Encodes all Unicode characters
 - Uses 1 byte for ASCII, but multiple bytes for other characters



Codes

Decimal	BCD	Excess-3	84-2-1	2421	Bi-quinary
algits	(8421)				(3043210)
0	0000	0011	0000	0000	0100001
1	0001	0100	0111	0001	0100010
2	0010	0101	0110	0010	0100100
3	0011	0110	0101	0011	0101000
4	0100	0111	0100	0100	0110000
5	0101	1000	1011	1011	1000001
6	0110	1001	1010	1100	1000010
7	0111	1010	1001	1101	1000100
8	1000	1011	1000	1110	1001000
9	1001	1100	1111	1111	1010000

Floating Point



(1.13)

Floating Point Numbers

- N = M \times r^E
 - M (mantissa or significand)
 - Significant digits of N
 - E (exponent or characteristic)
 - An integer exponent
- N = ± (a_{n-1} ... a₀ .a₋₁ ... a_{-m})_r
 - $\,\circ\,$ Presented by: N = \pm (.a_{n\text{-1}} ... $a_{\text{-m}})_r \times r^n$



Floating Point Numbers

• N = M \times r ^E	(1.13)
 M (mantissa or significand) 	(1110)
 Significant digits of N 	
 E (exponent or characteristic) 	
 An integer exponent 	
• M	
 Usually represented in sign magnitude 	
• $M = (S_{M}.a_{n-1}a_{-m})_{rsm}$	(1.14)
 (.a_{n-1} a_{-m})_r represents the magnitude 	
• $M = (-1)^{S_M} \times (.a_{n-1}a_{-m})_r$ (0: positive, 1: negative)	(1.15)



Floating Point Numbers

- $N = M \times r^E$
 - M (mantissa or significand)
 - Significant digits of N
 - E (exponent or characteristic)
 - An integer exponent

•
$$M = (S_M.a_{n-1}...a_{-m})_{rsm} = (-1)^{S_M} \times (.a_{n-1}...a_{-m})_r$$

(1.13)

(1.14), (1.15)

• E

- \circ -2^{e-1} \leq E \leq 2^{e-1}
- Usually coded in excess-K two's complement.
- K : called bias
 - Usually selected to be 2^{e-1} (e is the number of bits).
- Biased E is:
 - $\circ 0 \leq E + 2^{e-1} \leq 2^{e}$
- Excess-K form of E is written as
 - $E = (b_{e-1}, b_{e-2} \dots b_0)_{excess-K}$
 - b_{e-1} is the sign bit.

(1.16)



(1.13)

(1.16)

(1.17)

Floating Point Numbers • N = M \times r^E M (mantissa or significand) Significant digits of N • E (exponent or characteristic) An integer exponent • $M = (S_M.a_{n-1}...a_{-m})_{rsm} = (-1)^{S_M} \times (.a_{n-1}...a_{-m})_r$ (1.14), (1.15)• $E = (b_{e-1}, b_{e-2} \dots b_0)_{excess-K}$

• Combining Eqs. (1.14) and (1.16), we have • $N = (S_M b_{e-1} b_{e-2} \dots b_0 a_{n-1} \dots a_{-m})_r$

•
$$\mathbf{N} = (-1)^{S_M} \times (.a_{n-1}...a_{-m})_r \times r^{(b_{e-1}b_{e-2}...b_0)-2^{e-1}}$$
 (1.18)

• Number 0 is represented by an all-zero word.



	•	Multiple	representations	of a	a given	number:
--	---	----------	-----------------	------	---------	---------

• $N = M \times r^{E}$	(1.19)
• = (M \div r) \times r ^{E+1}	(1.20)
• = (M × r) × r ^{E-1}	(1.21)

- Example: M = +(1101.0101)₂
 - M = +(1101.0101)₂
 - = $(0.11010101)_2 \times 2^4$
 - = (0.011010101)₂ × 2⁵
 - = $(0.0011010101)_2 \times 2^6$



Normalization

- A unique representation
- Mantissa has a nonzero value in its MSD position.
- Example:
 - M = +(1101.0101)₂
 - $^\circ~$ Normal representation: (0.11010101)_2 $\times~2^4$



- Floating-point number formats
 - Typical single-precision format

•
$$\mathbf{N} = (\mathbf{S}_{\mathbf{M}}\mathbf{b}_{e-1}\mathbf{b}_{e-2} \dots \mathbf{b}_{0}\mathbf{a}_{n-1} \dots \mathbf{a}_{-m})_{r} (-1)^{S_{M}} \times (.a_{n-1}...a_{-m})_{r} \times r^{(b_{e-1}b_{e-2}...b_{0})-2^{e-1}}$$



• Typical extended-precision format

S _M	Exponent E	Mantissa M (most significant part)
----------------	------------	------------------------------------

Mantissa M (least significant part)



- • $N = (101101.101)_2$
- Assumption
 - Normalized sign magnitude fraction is used for M
 - Excess-16 two's complement is used for *E*.





•N = $(101101.101)_2$

- N = $(101101.101)_2 = (0.101101101)_2 \times 2^6$
- $M = +(0.1011011010)_2 = (0.1011011010)_{2sm}$
- $E = +(6)_{10} = +(0110)_2 = (00110)_{2cns}$
 - Add the bias $16 = (10000)_2$ to E
 - E = 00110 + 10000 = 10110
 - E = (1, 0110)_{excess-16}
- Combining M and E

N = (0, 10110, 101101101)_{fp}





System/ Format	Total bits	Significand bits	Exponent bits	Exponent blas	Mantissa coding
IEEE Std. 754-1985:					Sign/Mag: (radix 2):
Single Precision	32	23 (+1)	8	127	$1 \leq M < 2$
Double Precision	64	52 (+1)	11	1023	$1 \leq M < 2$
IBM System/360:					Sign/Mag (radix 16):
Single Precision	32	24	7	64	$1/16 \le M < 1$
Double Precision	64	56	7	64	$1/16 \le M < 1$
DEC VAX 11/780:					Sign/Mag (radix 2):
F Format	32	23 (+1)	8	128	$1/2 \le M < 1$
D Format	64	55 (+1)	8	128	$1/2 \le \boldsymbol{M} < 1$
G Format	64	52 (+1)	11	1024	$1/2 \le \boldsymbol{M} < 1$
CDC Cyber 70:	60	48	11	1024	1's Complement (radix 2)
					$1 \leq M < 2^{48}$



IEEE 754



IEEE 754 Special Number Representation

Single F	Precision	Double I	Precision Numb	er Represented			
Exponent	Significand	Exponent	Significand				
0	0	0	0	0			
0	nonzero	0	nonzero	Denormalized number			
1 to 254	anything	1 to 2046	anything	Floating Point Number			
255	0	2047	0	Infinity			
255	nonzero	2047	nonzero	NaN (Not A Number)			

Error Correction/Detection Codes



Error

• Error

An incorrect value in one or more bits

• Single error

• An incorrect value in only one bit

• Multiple error

• One or more bits are incorrect

• Error sources

- Hardware failures
- External interference (noise)
- Other unwanted events.





Error Detection/Correction Codes



- Error detection/correction code
 - Encode information in such a way that a particular class of errors can be detected and/or corrected.
- Let I and J be n-bit binary information words
 - w(I): number of 1's in I (weight)
 - d(I, J): number of bit positions in which I and J differ (distance)
- Example: I = (01101100) and J = (11000100)
 - w(I) = 4 and w(J) = 3
 - o d(I, J) = 3
 0 1 1 0 1 1 0 0
 1 1 0 0 1 0 0
 1 1 0 0 1 0 0
 ↑ ↑ ↑



- Minimum distance, d_{min} , of a code C
 - $^{\circ}~$ For any two code words I and J in C, $~d(I,\,J) \geq d_{min}$
 - Determines the properties of error correction and detection of a code





- \bullet Relationship between d_{\min} of code words and detection/correction ability
 - A code provides *t* error correction
 - Plus *s* error detection of *s* additional errors if and only if.





• Example

- Single-error detection (SED): s = 1, t = 0, $d_{min} = 2$.
- Single-error correction (SEC): $s = 0, t = 1, d_{min} = 3$.
- Single-error correction and double-error detection (SEC/DED)





- Simple Parity code
- Two-out-of-Five code
- Hamming code



Parity

- Simple Parity Code
 - Concatenate (|) a parity bit, *P*, to each code word of *C*.
 - Odd-parity code: w(P|C) is odd.
 - Even-parity code: w(P|C) is even.





• Sample

• Parity coding on magnetic tape:





• Produce odd parity code for ASCII code of character 0,X,=, BEL

Character	ASCII Code	
0	0110000	
Х	1011000	
=	0111100	
BEL	0000111	





• Produce odd parity code for ASCII code of character 0,X,=, BEL

Character	ASCII Code	Odd-parity Code
0	0110000	10110000
Х	1011000	01011000
=	0111100	1111100
BEL	0000111	00000111



- Error detection
 - Check whether a code word has the correct parity
- Does parity have error detection ability?



- Error detection
 - Check whether a code word has the correct parity
- Does parity have error detection ability?
 - Yes,
 - Single-error detection code ($d_{\min} = 2$).



Two-out-of-Five Code

- Two-out-of-Five code
 - Each code word has exactly two 1's and three 0's.
- Error detection
 - Counting the number of ones
 - If number of ones is not exactly equal to 2
 - $\bullet \rightarrow \text{error}$
 - Detects single and multiple errors in adjacent bits.

Digit	Two-out-of-Five Code
0	00011
1	00101
2	01001
3	10001
4	00110
5	01010
6	10010
7	01100
8	10100
9	11000



Hamming Codes

- Richard Hamming, 1950
- An extension of simple parity codes with multiple parity or check bits
- Each check bit
 - Is defined over (or covers) a subset of the information bits.

Subsets overlap

- Each information bit is in at least two subsets.
- Error detection/correction ability
 - Number of check bits
 - How check bits are defined
 - d_{min} : weight of the minimum-weight nonzero code word.



Hamming Codes (cont'd)

- Hamming Code 1 (Table 1.14)
 - A code word consists of 4 information bits and 3 check bits:
 - $c = (i_3 \, i_2 \, i_1 \, i_0 \, c_2 \, c_1 \, c_0)$
 - Each check bit covers:

 $C_{2}: i_{3}, i_{2}, i_{1}$ $C_{1}: i_{3}, i_{2}, i_{0}$ $C_{0}: i_{3}, i_{1}, i_{0}$

• d_{min} = 3

• Single error correction code.



Har	Hamming Codes (cont'd)										Iran Univer	sity of Science & T
• m =8				(. 1		or.				
• r?				(m	+r	+ 1	$) \leq 2$	<u>.</u>				
Bit position:	1	2	3	4	5	6	7	8	9	10	11	12
	P_1	P_2	1	P_4	1	0	0	P_8	0	1	0	0

 $P_{1} = \text{XOR of bits } (3, 5, 7, 9, 11) = 1 \oplus 1 \oplus 0 \oplus 0 \oplus 0 = 0$ $P_{2} = \text{XOR of bits } (3, 5, 7, 10, 11) = 1 \oplus 0 \oplus 0 \oplus 1 \oplus 0 = 0$ $P_{4} = \text{XOR of bits } (5, 6, 7, 12) = 1 \oplus 0 \oplus 0 \oplus 0 = 1$ $P_{8} = \text{XOR of bits } (9, 10, 11, 12) = 0 \oplus 1 \oplus 0 \oplus 0 = 1$

- $C_1 = XOR \text{ of bits}(1, 3, 5, 7, 9, 11)$
- $C_2 = \text{XOR of bits}(2, 3, 6, 7, 10, 11)$
- $C_4 = \text{XOR of bits}(4, 5, 6, 7, 12)$
- $C_8 = \text{XOR of bits} (8, 9, 10, 11, 12)$



Sample 1

- Send this data in haming code
- 10101001





• Find r?

 $10101001 \qquad (m+r+1) \le 2^r$ $r = 1 \quad \therefore \quad (8+1+1) \le 2^1 \qquad 10 \le 2$ $r = 2 \quad \therefore \quad (8+2+1) \le 2^2 \quad 11 \le 4$ $r = 3 \quad \therefore \quad (8+3+1) \le 2^3 \quad 12 \le 8$ $r = 4 \quad \therefore \quad (8+4+1) \le 2^4 \quad 13 \le 16$



- Determine P_i ? 1 2 3 4 5 6 7 8 9 10 11 12 ? ? 1 ? 0 1 0 ? 1 0 0 1
- $3=1+2 = 2^{0}+2^{1}$
- $5=1+4 = 2^{0}+2^{2}$
- $6=2+4 = 2^1+2^2$
- $7=1+2+4 = 2^0+2^1+2^2$
- $9=1+8 = 2^0+2^3$
- $10=2+8 = 2^1+2^3$
- $11=1+2+8 = 2^0+2^1+2^3$
- $12=4+8 = 2^2+2^3$



 • Determine P_i ?
 1 2 3 4 5 6 7 8 9 10 11 12

 ? ? 1 ? 0 1 0 ? 1 0 0 1

•
$$P_1 = xor(3,5,7,9,11) = xor(1,0,0,1,0) = 0$$

•
$$P_2 = xor(3,6,7,10,11) = xor(1,1,0,0,0) = 0$$

•
$$P_4 = xor(5,6,7,12) = xor(0,1,0,1) = 0$$

•
$$P_8 = xor(9,10,11,12) = xor(1,0,0,1) = 0$$



- $C_1 = xor(1,3,5,7,9,11) = xor(0,1,0,0,1,0) = 0$
- $C_2 = xor(2,3,6,7,10,11) = xor(0,1,1,0,0,0) = 0$
- $C_4 = xor(4,5,6,7,12) = xor(0,0,1,0,1) = 0$
- $C_8 = xor(8,9,10,11,12) = xor(0,1,0,0,1) = 0$



- $C_1 = xor(1,3,5,7,9,11) = xor(0,0,0,0,1,0) = 1$
- $C_2 = xor(2,3,6,7,10,11) = xor(0,0,1,0,0,0) = 1$
- $C_4 = xor(4,5,6,7,12) = xor(0,0,1,0,1) = 0$
- $C_8 = xor(8,9,10,11,12) = xor(0,1,0,0,1) = 0$



Thank You

