Digital Logic Design

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How to Find Equivalent States?

- Inspection
- Partitioning
- Implication Tables

Outline

State Reduction in Incompletely Specified Circuits

- State Compatibility
- Merger Diagrams
- Minimization Procedure

Simplification + Don’t Cares!

Simplifications + d

Next states and/or outputs are not specified for all states

How to optimize these sequential circuits?

Sample 1

Minimize the following incompletely specified circuit

	$x = 0$	$x = 1$
A	$B/—$	$E/0$
B	$B/1$	$E/—$
C	$F/0$	$C/0$
D	$B/1$	$A/1$
E	$D/0$	$C/—$
F	$D/—$	$C/1$

Minimizing Sample1

- The 4 don’t cares can be assigned in any way that we choose
- Once don’t cares are specified, the table is converted to completely specified table
- Next, the state reductions already we discussed can be applied here.
- Suppose we assign don’t cares in a such way that A becomes equivalent to B and E equivalent to F.

	$x=0$	$x=1$
A	$B/-$	$E/0$
B	$B/1$	$E/-$
C	$F/0$	$C/0$
D	$B/1$	$A/1$
E	$D/0$	$C/-$
F	$D/-$	$C/1$

The reduced state table:

	$x=0$	$x=1$
A'	$A'/1$	$D'/0$
B'	$D'/0$	$B'/0$
C'	$A'/1$	$A'/1$
D'	$C'/0$	$B'/1$

Minimizing Sample 1 (cont'd)

- A better assigning of the don't cares
- Don't cares of states A and E are assigned zeros and don't cares of B and F are assigned ones
- A, C and E become equivalent
- B, D and F become equivalent

	$x=0$	$x=1$
A	$B/—$	$E/0$
B	$B/1$	$E/—$
C	$F/0$	$C/0$
D	$B/1$	$A/1$
E	$D/0$	$C/—$
F	$D/—$	$C/1$

The reduced state table:

	$x=0$	$x=1$
A'	$B'/0$	$A'/0$
B'	$B'/1$	$A'/1$

The optimum simplification for don't cares is not obvious and simple and we should look for a solution

State Compatibility

Applicable Input Sequence

- *Applicable input sequences to an state S_i* :
  - When the circuit is in state S_i and the input sequence is applied, all next states **are specified** except possibly the last element of next-state sequence.

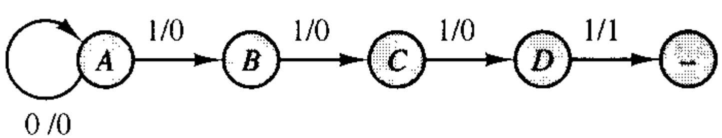
- Example:

	$x = 0$	$x = 1$
A	$A/0$	$B/0$
B	–/–	$C/0$
C	–/–	$D/0$
D	–/–	–/1

0111 is applicable to state A

1111 is applicable to state A

11111 is NOT applicable to state A



State Compatibility

- *Compatible states*:
  - Two states S_i and S_j are **compatible** if for **each input sequence applicable** to both states **the same output sequence** will be produced, when the outputs are specified.

- *Compatible states*:
  - Two states S_i and S_j are compatible if and only if the following conditions are satisfied for **any possible input I_p**
    - **Outputs** produced by S_i and S_j are the **same**, when both are specified
    - Next states S_k and S_l are compatible, when both are specified

State Compatibility: Sample 2

Determine state compatibility in the following state table

Input:		1		1		1		1	
State:	A		E		C		C		C
Output:		0		–		0		0	
State:	C		C		C		C		C
Output:		0		0		0		0	

- States A and C are **compatible**
- States A and E are **compatible**
- States C and E are **compatible**
- Finally ACE are **maximal compatible**

Compatibility Relations

- *Compatibility relation*:
  - Let ***R*** be a relation on a set ***S***
  - ***R*** is a compatibility relation on ***S*** if and only if it is reflexive and symmetric
  - A compatibility relation on a set partitions the set into compatibility classes
  - They are typically not disjoint.
- *Example*:
  - $S = \{A,B,C,D,E\}$
  - $R = \{(A,A),(B,B),(C,C),(D,D),(E,E),(A,B),(B,A),(A,C),(C,A), (A,D), (D,A), (A,E), (E,A), (B,D), (D,B), (C,D), (D,C), (C,E), (E,C)\}$
  - Compatibility classes are $(AB)(AC)(AD)(AE)(BD)(CD)(CE)(ABD)(ACD)(ACE)$
  - Incompatibility classes are $(BC)(BE)(DE)$
- Compatible pairs may be found using implication tables
- Maximal compatibles may be found using merger diagrams

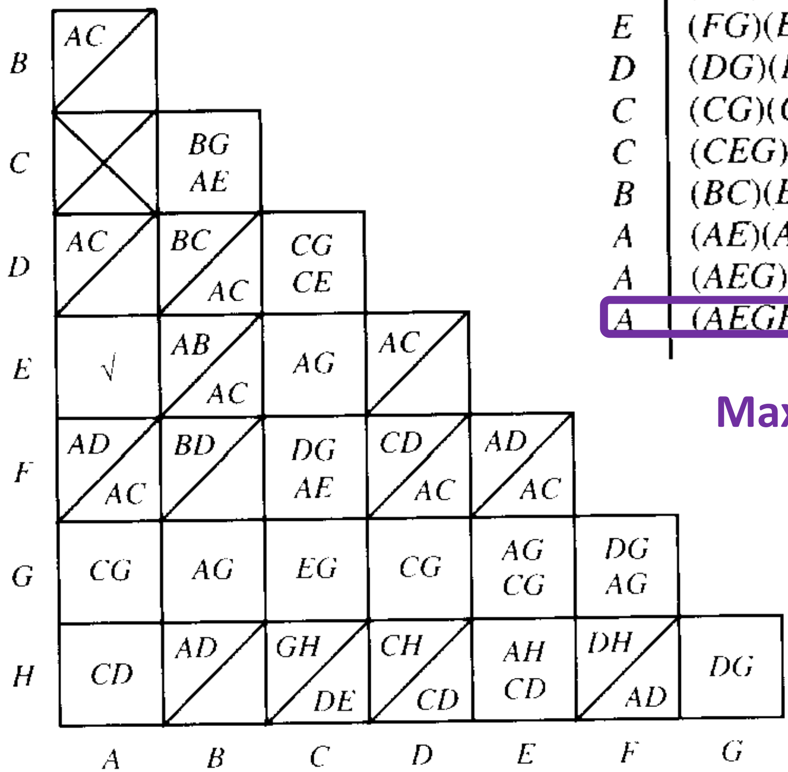
Compatible Classes: Sample 3

Determine the **compatible classes** for the following state table

	$x=0$	$x=1$
A	$A/-$	$C/1$
B	$B/-$	$A/-$
C	$G/-$	$E/0$
D	$C/1$	$C/-$
E	$A/1$	$C/-$
F	$D/-$	$A/-$
G	$G/-$	$G/-$
H	$H/-$	$D/-$

Compatible Classes: Sample 3

	x = 0	1
A	A/–	C/1
B	B/–	A/–
C	G/–	E/0
D	C/1	C/–
E	A/1	C/–
F	D/–	A/–
G	G/–	G/–
H	H/–	D/–



Compatible classes

G	(GH)
F	(GH)(FG)
E	(EG)(EH)(GH)(FG)
E	(FG)(EGH)
D	(DG)(FG)(EGH)
C	(CG)(CF)(CE)(CD)(DG)(FG)(EGH)
C	(CEG)(CDG)(CFG)(EGH)
B	(BC)(BG)(CEG)(CDG)(CFG)(EGH)
A	(AE)(AG)(AH)(BC)(BG)(CEG)(CDG)(CFG)(EGH)
A	(AEG)(AGH)(AEH)(BCG)(CEG)(CDG)(CFG)(EGH)
A	(AEGH)(BCG)(CDG)(CEG)(CFG)

Maximal compatible classes

Compatible Vs. Equivalence States

- What are the differences between compatibility in incompletely specified circuits and equivalence in completely specified circuits?
  - Transitive relation is hold in the completely specified circuits
  - Transitive relation does not work with the incompletely specified circuits

- Completely specified circuits:
  - If **A** is equivalent with **B** and **B** is equivalent with C we can conclude **A** is equivalent with C. → **A** and **B** and **C** are equivalence

- Incompletely specified circuits, there is **no such guarantee**
  - To have a **maximal compatibility** for example (BCG) we should have all pairs; i.e., **(B,C)** and **(C,G)** and **(B,G)**

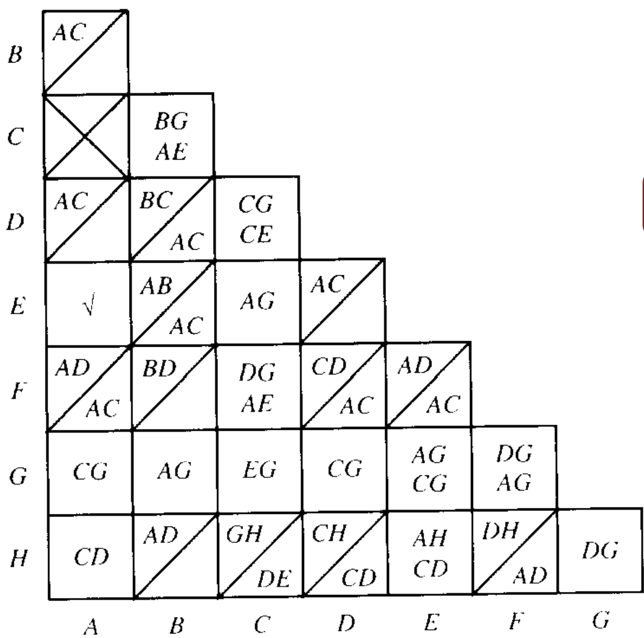
Incompatibility Classes: Sample 4

Determine the incompatibility classes for the following state table

	$x=0$	$x=1$
A	A/–	C/1
B	B/–	A/–
C	G/–	E/0
D	C/1	C/–
E	A/1	C/–
F	D/–	A/–
G	G/–	G/–
H	H/–	D/–

Incompatibility Classes: Sample 4 (cont’d)

	x = 0	1
A	A/–	C/1
B	B/–	A/–
C	G/–	E/0
D	C/1	C/–
E	A/1	C/–
F	D/–	A/–
G	G/–	G/–
H	H/–	D/–



The incompatible classes

G	–
F	(FH)
E	(FH)(EF)
D	(FH)(EF)(DH)(DF)(DE)
D	(FH)(DH)(DEF)
C	(CH)(FH)(DH)(DEF)
B	(BH)(BF)(BE)(BD)(CH)(FH)(DH)(DEF)
B	(BH)(BDEF)(CH)(FH)(DH)
B	(BDEF)(CH)(BDFH)
A	(AB)(AC)(AD)(AF)(BDEF)(CH)(BDFH)
A	(ABDF)(AC)(BDEF)(CH)(BDFH)

Maximal compatible classes

Merger Diagram

State Compatibility & Implication Table

- Implication table is **useful** in finding
  - **C**ompatible states
  - Maximal compatible
  - Incompatible classes
- Tedious procedure
- How about considering another tool?
  - **Merger Diagram, a graphical method!**

Merger Diagram Construction

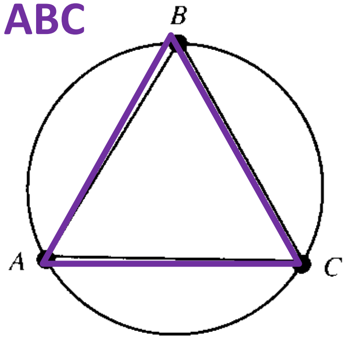
- Shows **states** by **dots** on a **circle**
  - Visually noting those sets in which every state is connected to every other state.
  - The maximal sets form regular graphical patterns.

- Find Maximal compatible
  - Connects each related (compatible) pairs of states

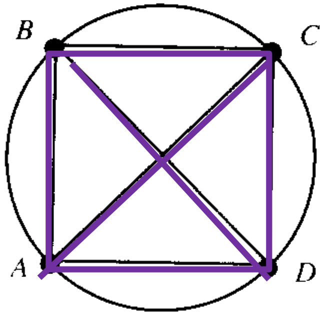
- Find incompatible classes
  - Connects each related (incompatible) pairs of states

Samples of Maximal Compatible Sets

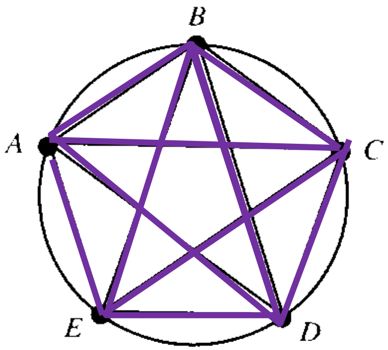
ABC



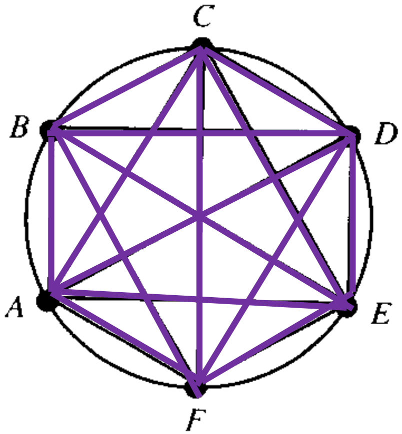
ABCD



ABCDE



ABCDEF



Merger Diagram Rules

- Let's define some **rules** to extract maximal sets from a merger diagram
- **Rule 1:**
  - Make each maximal set **as large as possible**
- **Rule 2**:
  - **Each state must** be interconnected with **every other state** in the maximal set
- **Rule 3:**
  - **Each related pair of states must** appear in **at least one maximal set**

Sample 5

- Draw the **merger diagram**
- Find maximal compatible class
- Find maximal incompatible class

	$x = 0$	$x = 1$
A	A/–	C/1
B	B/–	A/–
C	G/–	E/0
D	C/1	C/–
E	A/1	C/–
F	D/–	A/–
G	G/–	G/–
H	H/–	D/–

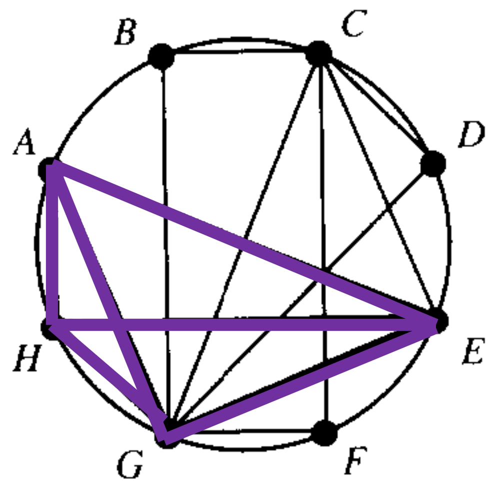
Sample 5: Implication Table

	x = 0	x = 1
A	A/–	C/1
B	B/–	A/–
C	G/–	E/0
D	C/1	C/–
E	A/1	C/–
F	D/–	A/–
G	G/–	G/–
H	H/–	D/–

	A	B	C	D	E	F	G
B	~~AC~~						
C		BG AE					
D	~~AC~~	~~BC AC~~	CG CE				
E	√	~~AB AC~~	AG	~~AC~~			
F	~~AD AC~~	~~BD~~	DG AE	~~CD AC~~	~~AD AC~~		
G	CG	AG	EG	CG	AG CG	DG AG	
H	CD	~~AD~~	~~GH DE~~	~~CH CD~~	AH CD	~~DH AD~~	DG

Sample 5: Merger Diagram: Compatible Classes

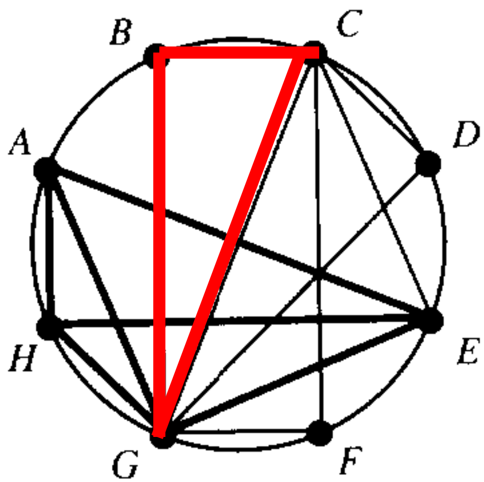
	A	B	C	D	E	F	G
B	~~AC~~						
C		BG AE					
D	~~AC~~	~~BC AC~~	CG CE				
E	√	~~AB AC~~	AG	~~AC~~			
F	~~AD AC~~	~~BD~~	DG AE	~~CD AC~~	~~AD AC~~		
G	CG	AG	EG	CG	AG CG	DG AG	
H	CD	~~AD~~	~~GH DE~~	~~CH CD~~	AH CD	~~DH AD~~	DG



- **AEGH**

Sample 5: Merger Diagram: Compatible Classes

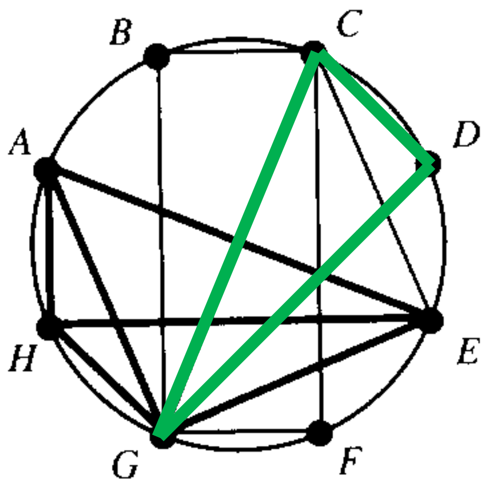
B	AC						
C	X	BG AE					
D	AC	BC AC	CG CE				
E	√	AB AC	AG	AC			
F	AD AC	BD	DG AE	CD AC	AD AC		
G	CG	AG	EG	CG	AG CG	DG AG	
H	CD	AD	GH DE	CH CD	AH CD	DH AD	DG
	A	B	C	D	E	F	G



- AEGH
- BCG

Sample 5: Merger Diagram: Compatible Classes

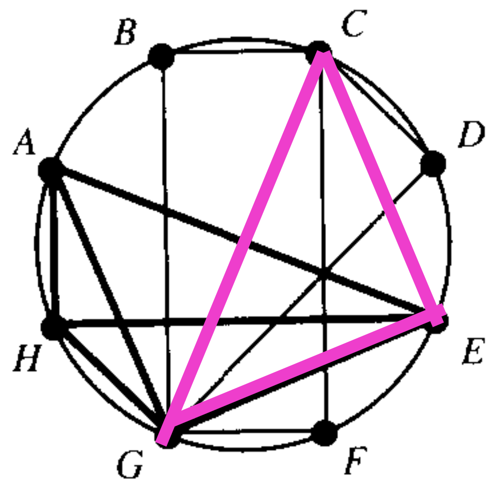
	A	B	C	D	E	F	G
B	~~AC~~						
C		BG AE					
D	~~AC~~	~~BC AC~~	CG CE				
E	√	~~AB AC~~	AG	~~AC~~			
F	~~AD AC~~	~~BD~~	DG AE	~~CD AC~~	~~AD AC~~		
G	CG	AG	EG	CG	AG CG	DG AG	
H	CD	~~AD~~	~~GH DE~~	~~CH CD~~	AH CD	~~DH AD~~	DG



- AEGH
- BCG
- CDG
- CEG
- CFG

Sample 5: Merger Diagram: Compatible Classes

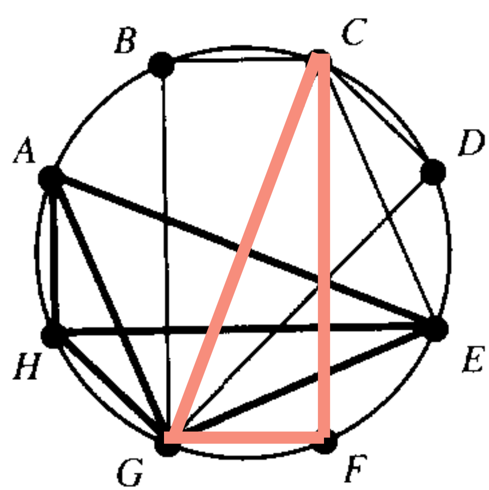
B	AC						
C		BG AE					
D	AC	BC AC	CG CE				
E	√	AB AC	AG	AC			
F	AD AC	BD	DG AE	CD AC	AD AC		
G	CG	AG	EG	CG	AG CG	DG AG	
H	CD	AD	GH DE	CH CD	AH CD	DH AD	DG
	A	B	C	D	E	F	G



- AEGH
- BCG
- CDG
- CEG

Sample 5: Merger Diagram: Compatible Classes

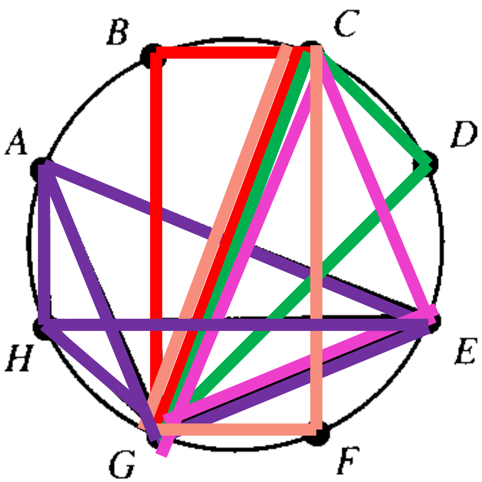
B	AC						
C	×	BG AE					
D	AC	BC AC	CG CE				
E	√	AB AC	AG	AC			
F	AD AC	BD	DG AE	CD AC	AD AC		
G	CG	AG	EG	CG	AG CG	DG AG	
H	CD	AD	GH DE	CH CD	AH CD	DH AD	DG
	A	B	C	D	E	F	G



- AEGH
- BCG
- CDG
- CEG
- CFG

Sample 5: Merger Diagram: Compatible Classes

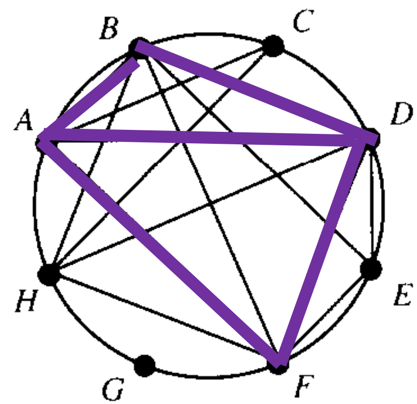
	A	B	C	D	E	F	G
B	AC						
C		BG AE					
D	AC	BC AC	CG CE				
E	√	AB AC	AG	AC			
F	AD AC	BD	DG AE	CD AC	AD AC		
G	CG	AG	EG	CG	AG CG	DG AG	
H	CD	AD	GH DE	CH CD	AH CD	DH AD	DG



- AEGH
- BCG
- CDG
- CEG
- CFG

Sample 5: Merger Diagram: Incompatible Classes

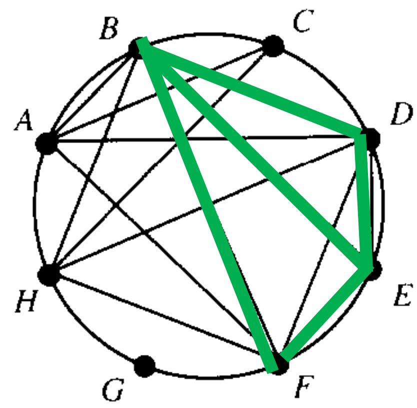
	A	B	C	D	E	F	G
B	~~AC~~						
C		BG AE					
D	~~AC~~	~~BC AC~~	CG CE				
E	√	~~AB AC~~	AG	~~AC~~			
F	~~AD AC~~	~~BD~~	DG AE	~~CD AC~~	~~AD AC~~		
G	CG	AG	EG	CG	AG CG	DG AG	
H	CD	~~AD~~	~~GH DE~~	~~CH CD~~	AH CD	~~DH AD~~	DG



- **ABDF**

Sample 5: Merger Diagram: Incompatible Classes

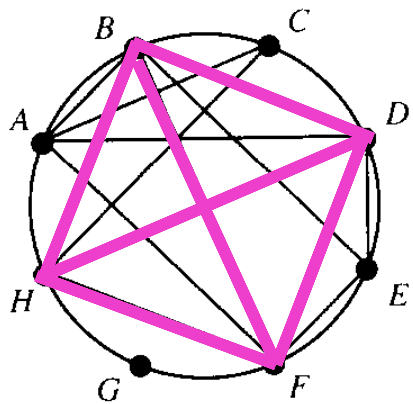
	A	B	C	D	E	F	G
B	~~AC~~						
C		BG AE					
D	~~AC~~	~~BC AC~~	CG CE				
E	√	~~AB AC~~	AG	~~AC~~			
F	~~AD AC~~	~~BD~~	DG AE	~~CD AC~~	~~AD AC~~		
G	CG	AG	EG	CG	AG CG	DG AG	
H	CD	~~AD~~	~~GH DE~~	~~CH CD~~	AH CD	~~DH AD~~	DG



- ABDF
- BDEF

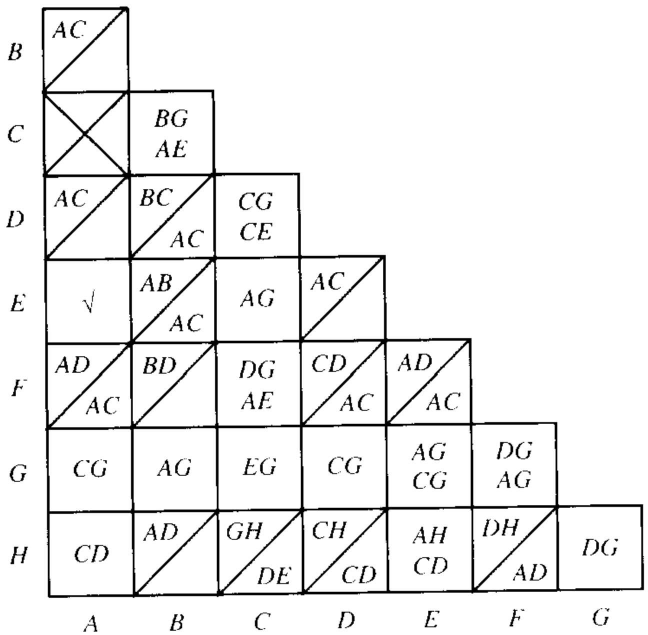
Sample 5: Merger Diagram: Incompatible Classes

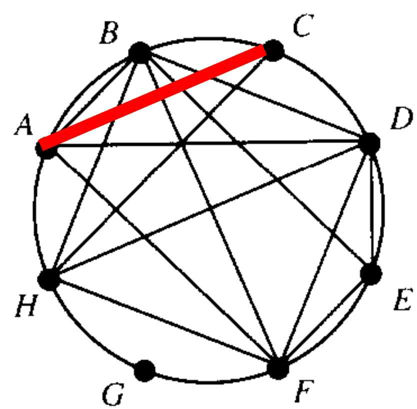
B	AC						
C		BG AE					
D	AC	BC AC	CG CE				
E	√	AB AC	AG	AC			
F	AD AC	BD	DG AE	CD AC	AD AC		
G	CG	AG	EG	CG	AG CG	DG AG	
H	CD	AD	GH DE	CH CD	AH CD	DH AD	DG
	A	B	C	D	E	F	G



- **ABDF**
- **BDEF**
- BDFH

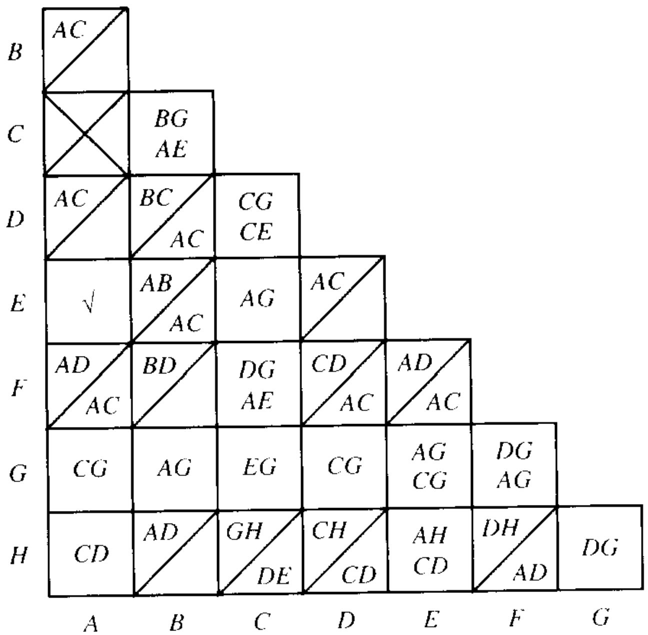
Sample 5: Merger Diagram: Incompatible Classes

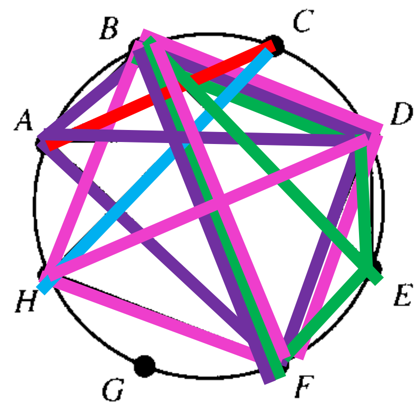




- ABDF
- BDEF
- BDFH
- AC

Sample 5: Merger Diagram: Incompatible Classes





- ABDF
- BDEF
- BDFH
- AC
- CH

Minimization Procedure

Minimization Procedure

- Selecting a set of compatible classes
- Each class in the set corresponds to a state in the reduced state table
- Finiah!

How to Find the a Set of Compatible Classes

- A set of compatible classes **must** meet **these three conditions**
  - **Completeness:** Union of all the sets **must** contain all states of the machine
  - **Consistency:** Chosen set of classes **must** be closed:
    - Implied next states of each class must be contained by some other classes in the set
  - **Minimality:** Smallest number of classes that meet the above two conditions

The procedure to find compatible classes that meet the above three conditions is a try and error technique.

To handle this, bound the number of states k required in the realization of the minimal state circuit.

Is this Procedure Effective?

- Finding compatible classes procedure that meet the three conditions is a **try and error** technique.

- To handle this, **bound the number of states k** required in the realization of the minimal state circuit.

Bounding Minimization Procedure

- Upper bound, U, of number of states in the minimal circuit
  - U = minimum{ **NSMC**, **NSOC**}
    - **NSMC**: **N**umber of **S**ets of **M**aximal **C**ompatible
    - **NSOC**: **N**umber of **S**tates of **O**riginal **C**ircuit
  - We do **not need more states** in the **minimal circuit** than the **number of states in the original circuit!**
- Lower bound, L, of number of states in the minimal circuit
  - L = maximum{ $NSMI_1$, $NSMI_2$, …, $NSMI_i$, …}
    - $NSML_i$: Number of Ntates in the i_{th} group of the sets of Maximal Incompatibles of the original circuit
  - If there are **two states** in the **original circuit** that are **incompatible**, **the minimal circuit** will **have to have at least two states** in order to **distinguish the incompatible ones.**

State Reduction Algorithms

State Reduction Algorithm

- Step 1: Find the **maximal compatible classes** using the **implication table** and **merger diagram**
- Step 2: Find the maximal incompatible classes
- Step 3: Find the **bounds** on the number of required states: U and L
- Step 4: Find a **set of compatible classes** that satisfy **completeness, consistency, and minimality**, by trial and error
- Step 5: Produce the **minimal State table** (it may still contain unspecified next states and outputs)

Sample 6

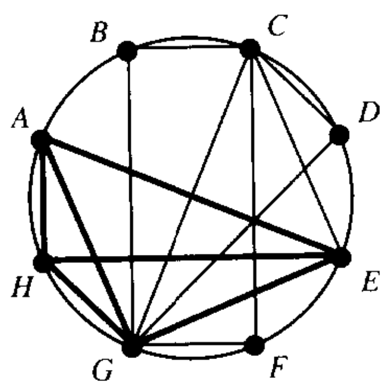
Find a **reduced state table** for the following state table.

	$x = 0$	$x = 1$
A	A/–	C/1
B	B/–	A/–
C	G/–	E/0
D	C/1	C/–
E	A/1	C/–
F	D/–	A/–
G	G/–	G/–
H	H/–	D/–

Sample 6: State Reduction Algorithm

	x = 0	1
A	A/–	C/1
B	B/–	A/–
C	G/–	E/0
D	C/1	C/–
E	A/1	C/–
F	D/–	A/–
G	G/–	G/–
H	H/–	D/–

	A	B	C	D	E	F	G
B	~~AC~~						
C	×	BG AE					
D	~~AC~~	~~BC AC~~	CG CE				
E	√	~~AB AC~~	AG	~~AC~~			
F	~~AD AC~~	~~BD~~	DG AE	~~CD AC~~	~~AD AC~~		
G	CG	AG	EG	CG	AG CG	DG AG	
H	CD	~~AD~~	~~GH DE~~	~~CH CD~~	AH CD	~~DH AD~~	DG



- **Maximal Compatible Classes**
  - **AEGH**
  - **BCG**
  - **CDG**
  - **CEG**
  - **CFG**

Sample 6: Closure Table

- Closure Table
  - Treating the maximal compatibles as states
  - Finding their sets for next states

- **Maximal Compatible Classes**
  - **AEGH**
  - **BCG**
  - **CDG**
  - **CEG**
  - **CFG**

	$x = 0$	$x = 1$
$(AEGH)$	AGH	CDG
(BCG)	BG	AEG
(CDG)	CG	CEG
(CFG)	DG	AEG
(CEG)	AG	CEG

Sample 6: Upper Bound

- Finding the upper bound (U)
  - **NSMC**: **N**umber of **S**ets of **M**aximal **C**ompatible
    - 5
  - **NSOC**: **N**umber of **S**tates of **O**riginal **C**ircuit
    - 8
  - U = min{5,8} = 5

	x = 0	x = 1
A	A/–	C/1
B	B/–	A/–
C	G/–	E/0
D	C/1	C/–
E	A/1	C/–
F	D/–	A/–
G	G/–	G/–
H	H/–	D/–

Maximal Compatible Classes

- **AEGH**
- **BCG**
- **CDG**
- **CEG**
- **CFG**

Sample 6: Lower Bound

- Finding the Lower bound (L)
  - $NSML_i$: Number of States in the i_{th} group of the sets of Maximal Incompatibles
  - Set of maximal incompatibles is
    - (ABDF)(BDEF)(BDFH)(AC)(CH)
    - L = max{**4**, **4**, **4**, **2**, **2**} = **4**

Maximal Incompatible Classes

- ABDF
- BDEF
- BDFH
- AC
- CH

Sample 6: Bounding

- #states in the reduction table is
  - $L \leq K \leq U$
  - $4 \leq K \leq 5$
- There are two options
  - $K = 4$
  - $K = 5$
- Which one works?

Sample 6: Closure Table (K=5)

	x = 0	1
A	A/–	C/1
B	B/–	A/–
C	G/–	E/0
D	C/1	C/–
E	A/1	C/–
F	D/–	A/–
G	G/–	G/–
H	H/–	D/–

- A' = (AEGH)
- B' = (BCG)
- C' = (CDG)
- D' = (CEG)
- E' = (CFG)

	x = 0	1
(AEGH)	AGH	CDG
(BCG)	BG	AEG
(CDG)	CG	CEG
(CFG)	DG	AEG
(CEG)	AG	CEG

	x = 0	1
A'	A'/1	C'/1
B'	B'/–	A'/0
C'	B', C', D', E'/1	D'/0
D'	A'/1	D'/0
E'	C'/–	A'/0

Next State

Sample 6: Reduce Table (K=4)

- No way to reduce states
- Four states of closure table should satisfies two conditions:
  - Completeness
  - Consistency
- A trial and error procedure shows that it is impossible for four states

Sample 6:

- Find the reduced table for the following state table

	$x = 0$	$x = 1$
A	$A/-$	$-/-$
B	$C/1$	$B/0$
C	$D/0$	$-/1$
D	$-/-$	$B/-$
E	$A/0$	$C/1$

Step 1: Find Maximal Compatible Set

- Implication table

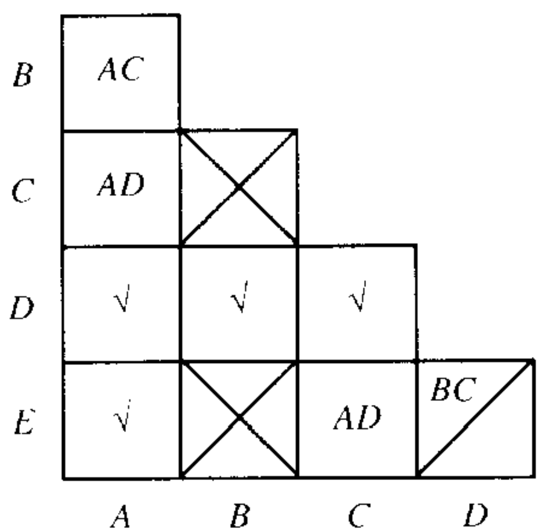
	x = 0	x = 1
A	A/–	–/–
B	C/1	B/0
C	D/0	–/1
D	–/–	B/–
E	A/0	C/1

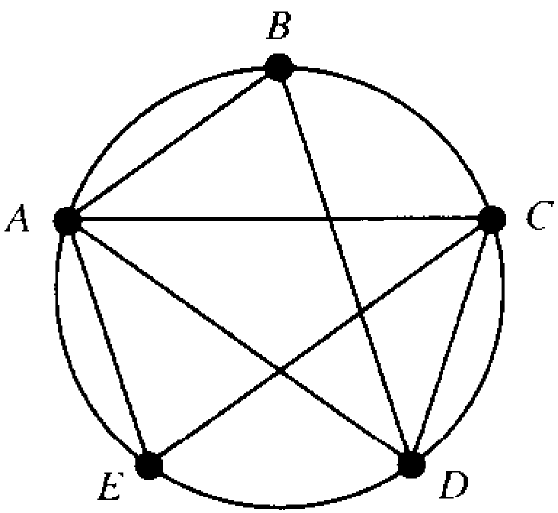
B	AC			
C	AD			
D	√	√	√	
E	√		AD	BC
	A	B	C	D

Step 1: Find Maximal Compatible Set (cont'd)

- Maximum Compatible Set
  - (ABD)
  - (ACD)
  - (ACE)

	x = 0	x = 1
A	A/–	–/–
B	C/1	B/0
C	D/0	–/1
D	–/–	B/–
E	A/0	C/1

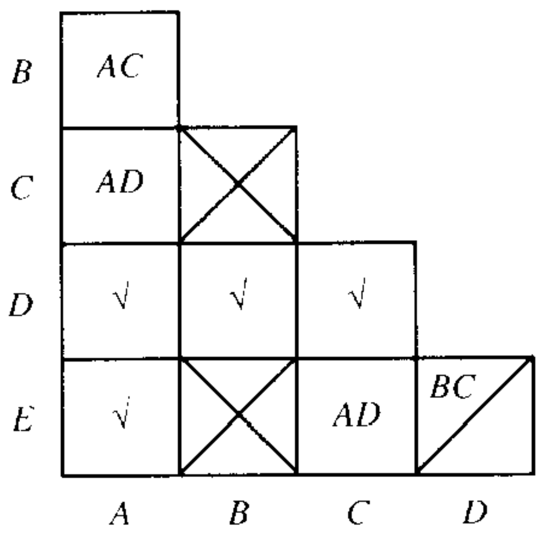


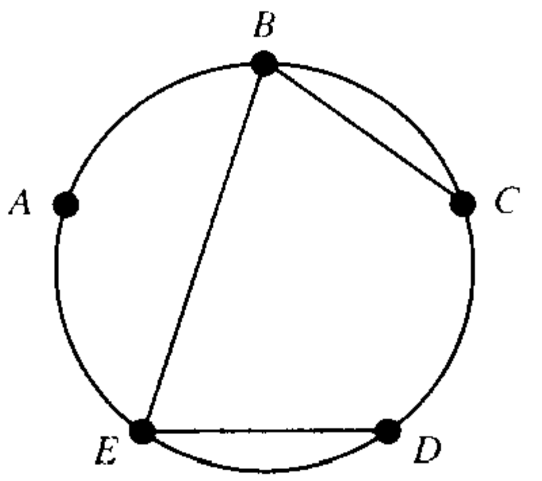


Step 2: Find Incompatible Set

- Maximum Incompatible Set
  - (BC)
  - (BE)
  - (DE)

	x = 0	x = 1
A	A/–	–/–
B	C/1	B/0
C	D/0	–/1
D	–/–	B/–
E	A/0	C/1





Step 3: Calculate Bounds

- Finding the upper bound (U)
  - **NSMC**: **N**umber of **S**ets of **M**aximal **C**ompatible
    - 3
  - **NSOC**: **N**umber of **S**tates of **O**riginal **C**ircuit
    - 5
  - $U = \min\{3,5\} = 3$
- Finding the Lower bound (L)
  - $NSML_i$: Number of States in the i_{th} group of the sets of Maximal Incompatibles
  - Set of maximal incompatibles is
    - (BC)(BE)(DE)
    - $L = \max\{2, 2, 2\} = 2$
- $2 \leq k \leq 3$

Step 4: Construct the Closure Table (k = 3)

- k = 2

	x = 0	1
A	$A/-$	$-/-$
B	$C/1$	$B/0$
C	$D/0$	$-/1$
D	$-/-$	$B/-$
E	$A/0$	$C/1$

- Maximum Compatible Set
  - (ABD)
  - (ACD)
  - (ACE)

	x = 0	1
(ABD)	AC	B
(ACD)	AD	B
(ACE)	AD	C

Step 5: Reduce the State Table

- k = 2
- Select the maximal compatibles (ABD) and (ACE)
  - Completeness: Covers all states of the original circuit
  - Consistency: Chosen set is close
    - Implied next states of each class must be contained by some other classes in the set
- A' = (ABD) and B' = (ACE)

	$x = 0$	$x = 1$
(ABD)	AC	B
(ACD)	AD	B
(ACE)	AD	C

	$x = 0$	$x = 1$
A'	B'/1	A'/0
B'	A'/0	B'/1

Thank You